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The degree of an eight-dimensional real quadratic division algebra is 1, 3, or 5

Ernst Dieterich ^{*}, Ryszard Rubinsztein

Matematiska Institutionen, Uppsala Universitet, Box 480, SE-751 06 Uppsala, Sweden

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Abstract

A celebrated theorem of Hopf (1940) [11], Bott and Milnor (1958) [1], and Kervaire (1958) [12] states that every finite-dimensional real division algebra has dimension 1, 2, 4, or 8. While the real division algebras of dimension 1 or 2 and the real quadratic division algebras of dimension 4 have been classified (Dieterich (2005) [6], Dieterich (1998) [3], Dieterich and Öhman (2002) [9]), the problem of classifying all 8-dimensional real quadratic division algebras is still open. We contribute to a solution of that problem by proving that every 8-dimensional real quadratic division algebra has degree 1, 3, or 5. This statement is sharp. It was conjectured in Dieterich et al. (2006) [7].

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1. Introduction

Let A be an algebra over a field k , i.e. a vector space over k equipped with a k -bilinear multiplication $A \times A \rightarrow A$, $(x, y) \mapsto xy$. Every element $a \in A$ determines k -linear operators $L_a : A \rightarrow A$, $x \mapsto ax$ and $R_a : A \rightarrow A$, $x \mapsto xa$. A *division algebra* over k is a non-zero k -algebra A such that L_a and R_a are bijective for all $a \in A \setminus \{0\}$. A *quadratic algebra* over k is a non-zero k -algebra A with unity 1, such that for all $x \in A$ the sequence $1, x, x^2$ is k -linearly dependent.

Here we focus on quadratic division algebras which are real (i.e. $k = \mathbb{R}$) and finite-dimensional. They form a category \mathcal{D}^q whose morphisms $f : A \rightarrow B$ are non-zero linear maps

^{*} Corresponding author.

E-mail addresses: Ernst.Dieterich@math.uu.se (E. Dieterich), Ryszard.Rubinsztein@math.uu.se (R. Rubinsztein).

satisfying $f(xy) = f(x)f(y)$ for all $x, y \in A$. For any positive integer n , the class of all n -dimensional objects in \mathcal{D}^q forms a full subcategory \mathcal{D}_n^q of \mathcal{D}^q . The (1, 2, 4, 8)-theorem for finite-dimensional real division algebras [11, 12] implies that $\mathcal{D}^q = \mathcal{D}_1^q \cup \mathcal{D}_2^q \cup \mathcal{D}_4^q \cup \mathcal{D}_8^q$. It is well known (see e.g. [9]) that both \mathcal{D}_1^q and \mathcal{D}_2^q consist of one isoclass only, represented by \mathbb{R} and \mathbb{C} , respectively. (Even the categories \mathcal{D}_1 and \mathcal{D}_2 of all 1-dimensional and 2-dimensional real division algebras respectively are classified. While $\mathcal{D}_1 = \mathcal{D}_1^q$ [9], \mathcal{D}_2 consists of a 4-parameter family of isoclasses [6].) The category \mathcal{D}_4^q is no longer trivial, but still its structure is well understood and its objects are classified by a 9-parameter family [3, 5, 9]. In contrast, the category \mathcal{D}_8^q seems to be much more difficult to approach. The problem of understanding its structure appears to have a very hard core, which still is far from a finishing solution. Among the insight so far obtained, the *degree* of an 8-dimensional real quadratic division algebra is notable. Following [7] it is a natural number $\deg(A)$ associated with any $A \in \mathcal{D}_8^q$, which is invariant under isomorphisms and satisfies the estimate $1 \leq \deg(A) \leq 5$. Examples of algebras $A \in \mathcal{D}_8^q$ having degree 1, 3, and 5 respectively are also given in [7]. For every $d \in \{1, \dots, 5\}$, the class of all algebras in \mathcal{D}_8^q having degree d forms a full subcategory \mathcal{D}_8^{qd} of \mathcal{D}_8^q .

In the present article we resume the investigation of $\deg(A)$. In Section 2 we apply topological arguments to prove that $\deg(A)$ always is odd. It follows that the category \mathcal{D}_8^q decomposes into three non-empty blocks \mathcal{D}_8^{q1} , \mathcal{D}_8^{q3} , and \mathcal{D}_8^{q5} . In Section 3 we summarize our structural insight into these three blocks, concluding that the problem of understanding the structure of \mathcal{D}_8^q has been reduced to the problem of understanding the structure of the category of all 7-dimensional *dissident algebras* having degree 3 or 5.

2. Main result

Towards our main result we need to recall a few facts related to the notion of a dissident map. Let V be a vector space over a field k . Following [8], a *dissident map* on V is a k -linear map $\eta : V \wedge V \rightarrow V$ such that $v, w, \eta(v \wedge w)$ are k -linearly independent whenever v, w are. It is well known that a finite-dimensional real vector space V admits a dissident map if and only if $\dim(V) \in \{0, 1, 3, 7\}$ [4, Proposition 7].

Let $\eta : \mathbb{R}^7 \wedge \mathbb{R}^7 \rightarrow \mathbb{R}^7$ be a dissident map. We equip \mathbb{R}^7 with the standard scalar product $\mathbb{R}^7 \times \mathbb{R}^7 \rightarrow \mathbb{R}$, $(v, w) \mapsto v^t w$. For every $v \in \mathbb{R}^7 \setminus \{0\}$, the subspace $\eta(v \wedge v^\perp) = \{\eta(v \wedge w) \mid w \in v^\perp\}$ in \mathbb{R}^7 is a hyperplane, which only depends on the line $[v]$ spanned by v . Thus η induces a map

$$\eta_{\mathbb{P}} : \mathbb{P}(\mathbb{R}^7) \rightarrow \mathbb{P}(\mathbb{R}^7), \quad \eta_{\mathbb{P}}([v]) = (\eta(v \wedge v^\perp))^\perp,$$

which actually is bijective [8, Proposition 2.2]. Following [7], a *lifting* of $\eta_{\mathbb{P}}$ is a map $\Phi : \mathbb{R}^7 \rightarrow \mathbb{R}^7$, $\Phi(v) = (\varphi_1(v), \dots, \varphi_7(v))$, satisfying the following three conditions: (a) all component maps $\varphi_1, \dots, \varphi_7$ are homogeneous real polynomials of common degree $d \geq 1$; (b) if $v \in \mathbb{R}^7 \setminus \{0\}$, then $\Phi(v) \neq 0$ and $[\Phi(v)] = \eta_{\mathbb{P}}([v])$; (c) the polynomials $\varphi_1, \dots, \varphi_7$ are relatively prime.

According to [7, Theorem 2.4] a lifting Φ of $\eta_{\mathbb{P}}$ exists, is unique up to non-zero real multiples, and satisfies $1 \leq d \leq 5$. It is therefore justified to define the *degree* of a dissident map η on \mathbb{R}^7 by $\deg(\eta) := \deg(\Phi) := d$.

Proposition 2.1. *The degree of any dissident map on \mathbb{R}^7 is odd.*

Proof. Our proof of Proposition 2.1 is topological in nature and, in particular, uses the fundamental group of the projective space $\mathbb{P}(\mathbb{R}^7)$.

Let $\pi : \mathbb{R}^7 \setminus \{0\} \rightarrow \mathbb{P}(\mathbb{R}^7)$ be the quotient map mapping a vector $v \in \mathbb{R}^7 \setminus \{0\}$ to the line $[v] \in \mathbb{P}(\mathbb{R}^7)$ spanned by v . We equip $\mathbb{R}^7 \setminus \{0\}$ with its standard Euclidean topology and $\mathbb{P}(\mathbb{R}^7)$ with the quotient topology. For a point $z \in \mathbb{P}(\mathbb{R}^7)$ let $\pi_1(\mathbb{P}(\mathbb{R}^7), z)$ be the fundamental group of $\mathbb{P}(\mathbb{R}^7)$ based at z . We recall that

$$\pi_1(\mathbb{P}(\mathbb{R}^7), z) = \mathbb{Z}_2,$$

$\mathbb{Z}_2 = \{0, 1\}$ being the group of integers modulo 2 [2, Section III.5].

The map $\pi : \mathbb{R}^7 \setminus \{0\} \rightarrow \mathbb{P}(\mathbb{R}^7)$ is a locally trivial fibration with the fibre homeomorphic to $\mathbb{R} \setminus \{0\}$. We choose a point $z \in \mathbb{P}(\mathbb{R}^7)$ and a vector $\tilde{z} \in \mathbb{R}^7 \setminus \{0\}$ such that $\pi(\tilde{z}) = z$. Then every vector w belonging to the fibre $\pi^{-1}(z)$ is of the form $w = q\tilde{z}$ with $q \in \mathbb{R}, q \neq 0$.

Denote by I the unit interval $[0, 1]$.

Let $\sigma : I \rightarrow \mathbb{P}(\mathbb{R}^7)$ be a loop in $\mathbb{P}(\mathbb{R}^7)$ at the point z i.e. σ is a continuous mapping such that $\sigma(0) = \sigma(1) = z$. Since $\pi : \mathbb{R}^7 \setminus \{0\} \rightarrow \mathbb{P}(\mathbb{R}^7)$ is a fibration, it follows from the Homotopy Lifting Theorem, [2, Theorem VII.6.4], that the loop $\sigma : I \rightarrow \mathbb{P}(\mathbb{R}^7)$ can be lifted to a continuous mapping $\tilde{\sigma} : I \rightarrow \mathbb{R}^7 \setminus \{0\}$ such that $\pi \circ \tilde{\sigma} = \sigma$ and $\tilde{\sigma}(0) = \tilde{z}$. (Observe that, in general, $\tilde{\sigma}$ will not be a loop but just a path.) Since $\sigma(1) = z$, it follows that $\tilde{\sigma}(1) \in \pi^{-1}(z)$ and, hence, that $\tilde{\sigma}(1) = q\tilde{z}$ for some $q = q(\sigma) \in \mathbb{R}, q \neq 0$. (Actually, the real number q depends not only on the loop σ but also on the choice of the lifting $\tilde{\sigma}$ which is not unique.) The next lemma is rather obvious. We include a proof for the convenience of the reader.

Lemma 2.2. *The loop $\sigma : I \rightarrow \mathbb{P}(\mathbb{R}^7)$ represents the trivial element in $\pi_1(\mathbb{P}(\mathbb{R}^7), z) = \mathbb{Z}_2$ if and only if $q > 0$.*

Proof. (\Leftarrow) Suppose that $q > 0$. Then there is a line segment $\tilde{\tau}$ in $\pi^{-1}(z)$ from $q\tilde{z}$ to \tilde{z} consisting of points of the form $s\tilde{z}$ with s between 1 and q . The path product $\bar{\sigma} = \tilde{\sigma} * \tilde{\tau}$ is now a loop in $\mathbb{R}^7 \setminus \{0\}$ starting and ending at \tilde{z} . The space $\mathbb{R}^7 \setminus \{0\}$ is homotopy equivalent to the 6-dimensional sphere S^6 and hence the fundamental group $\pi_1(\mathbb{R}^7 \setminus \{0\}, \tilde{z}) = 0$ is trivial. Therefore the loop $\bar{\sigma} = \tilde{\sigma} * \tilde{\tau}$ is null-homotopic in $\mathbb{R}^7 \setminus \{0\}$. It follows that the loop $\pi \circ \bar{\sigma} = (\pi \circ \tilde{\sigma}) * (\pi \circ \tilde{\tau}) = \sigma * (\pi \circ \tilde{\tau})$ is null-homotopic in $\mathbb{P}(\mathbb{R}^7)$. Since $\pi \circ \tilde{\tau}$ is a constant loop, the loops $\sigma * (\pi \circ \tilde{\tau})$ and σ are homotopic. Therefore the loop σ is null-homotopic in $\mathbb{P}(\mathbb{R}^7)$ and represents the trivial element in $\pi_1(\mathbb{P}(\mathbb{R}^7), z)$.

(\Rightarrow) Suppose that the loop $\sigma : I \rightarrow \mathbb{P}(\mathbb{R}^7)$ represents the trivial element in $\pi_1(\mathbb{P}(\mathbb{R}^7), z)$. Thus there exists a continuous mapping (homotopy) $H : I \times I \rightarrow \mathbb{P}(\mathbb{R}^7)$ such that $H(t, 0) = \sigma(t)$ and $H(0, s) = H(1, s) = H(t, 1) = z$ for all $t, s \in I$. Again, according to the Homotopy Lifting Theorem there exists a continuous mapping (lift) $\tilde{H} : I \times I \rightarrow \mathbb{R}^7 \setminus \{0\}$ such that $\pi \circ \tilde{H} = H$ and $\tilde{H}(t, 0) = \tilde{\sigma}(t)$, $\tilde{H}(0, s) = \tilde{z}$ and $\tilde{H}(1, s) = \tilde{\sigma}(1)$ for all $t, s \in I$. It follows, in particular, that $\tilde{H}(t, 1) \in \pi^{-1}(z)$ for all $t \in I$ and that $\tilde{H}(0, 1) = \tilde{z}$ while $\tilde{H}(1, 1) = \tilde{\sigma}(1) = q\tilde{z}$. In other words, $\tilde{H}(t, 1), t \in I$, is an arc in $\pi^{-1}(z)$ from \tilde{z} to $\tilde{\sigma}(1) = q\tilde{z}$. As $\pi^{-1}(z)$ consists of points of the form $r\tilde{z}$ with $r \in \mathbb{R} \setminus \{0\}$ it follows that $q > 0$.

That completes the proof of Lemma 2.1. \square

The diagram

$$\begin{array}{ccc} \mathbb{R}^7 \setminus \{0\} & \xrightarrow{\Phi} & \mathbb{R}^7 \setminus \{0\} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}(\mathbb{R}^7) & \xrightarrow{\eta_{\mathbb{P}}} & \mathbb{P}(\mathbb{R}^7) \end{array}$$

commutes. Since the components $\varphi_1, \dots, \varphi_7$ of Φ are polynomials, the map $\Phi : \mathbb{R}^7 \setminus \{0\} \rightarrow \mathbb{R}^7 \setminus \{0\}$ is continuous. It follows that the map $\eta_{\mathbb{P}} : \mathbb{P}(\mathbb{R}^7) \rightarrow \mathbb{P}(\mathbb{R}^7)$ is also continuous. The map $\eta_{\mathbb{P}}$ is bijective [8, Proposition 2.2] and the projective space $\mathbb{P}(\mathbb{R}^7)$ is compact and Hausdorff. Therefore $\eta_{\mathbb{P}}$ is a homeomorphism. Let us choose a point $z_0 \in \mathbb{P}(\mathbb{R}^7)$ and let us denote by z_1 the point $\eta_{\mathbb{P}}(z_0) \in \mathbb{P}(\mathbb{R}^7)$. The homeomorphism $\eta_{\mathbb{P}}$ induces a group isomorphism

$$\eta_{\mathbb{P}*} : \pi_1(\mathbb{P}(\mathbb{R}^7), z_0) \xrightarrow{\cong} \pi_1(\mathbb{P}(\mathbb{R}^7), z_1).$$

Let us choose a point $\tilde{z}_0 \in \pi^{-1}(z_0) \subset \mathbb{R}^7 \setminus \{0\}$. Let us denote by \tilde{z}_1 the point $\Phi(\tilde{z}_0) \in \mathbb{R}^7 \setminus \{0\}$. Then $\tilde{z}_1 = \Phi(\tilde{z}_0) \in \pi^{-1}(z_1)$.

Since the space $\mathbb{R}^7 \setminus \{0\}$ is path-connected, we can find a path $\tilde{\alpha} : I \rightarrow \mathbb{R}^7 \setminus \{0\}$ such that $\tilde{\alpha}(0) = \tilde{z}_0$ and $\tilde{\alpha}(1) = -\tilde{z}_0$. Then the composition $\alpha = \pi \circ \tilde{\alpha} : I \rightarrow \mathbb{P}(\mathbb{R}^7)$ is a loop in $\mathbb{P}(\mathbb{R}^7)$ which starts and ends at $z_0 = \pi(\tilde{z}_0) = \pi(-\tilde{z}_0)$. The path $\tilde{\alpha}$ is a lift of the loop α to $\mathbb{R}^7 \setminus \{0\}$ which starts at \tilde{z}_0 and ends at $-\tilde{z}_0$. Thus, according to Lemma 2.1, the loop α represents the non-trivial element of $\pi_1(\mathbb{P}(\mathbb{R}^7), z_0)$.

Let us now consider the path $\beta = \Phi \circ \tilde{\alpha} : I \rightarrow \mathbb{R}^7 \setminus \{0\}$ in $\mathbb{R}^7 \setminus \{0\}$ starting at $\tilde{z}_1 = \Phi(\tilde{z}_0) \in \pi^{-1}(z_1)$ and ending at $\tilde{z}_2 = \Phi(-\tilde{z}_0) \in \pi^{-1}(z_1)$. If $d = \deg(\Phi)$ is the degree of Φ then, by the definition of the degree, $\tilde{z}_2 = \Phi(-\tilde{z}_0) = (-1)^d \Phi(\tilde{z}_0) = (-1)^d \tilde{z}_1$.

The composition $\beta = \pi \circ \tilde{\beta} : I \rightarrow \mathbb{P}(\mathbb{R}^7)$ is a loop in $\mathbb{P}(\mathbb{R}^7)$ which starts and ends at the point $z_1 = \pi(\tilde{z}_1) = \pi(\tilde{z}_2)$. Moreover,

$$\beta = \pi \circ \tilde{\beta} = \pi \circ \Phi \circ \tilde{\alpha} = \eta_{\mathbb{P}} \circ \pi \circ \tilde{\alpha} = \eta_{\mathbb{P}} \circ \alpha.$$

Thus the homotopy class $[\beta]$ of the loop β in $\pi_1(\mathbb{P}(\mathbb{R}^7), z_1)$ is equal to $[\eta_{\mathbb{P}} \circ \alpha] = \eta_{\mathbb{P}*}[\alpha]$. As the homotopy class $[\alpha]$ of α in $\pi_1(\mathbb{P}(\mathbb{R}^7), z_0)$ was non-trivial and $\eta_{\mathbb{P}*} : \pi_1(\mathbb{P}(\mathbb{R}^7), z_0) \rightarrow \pi_1(\mathbb{P}(\mathbb{R}^7), z_1)$ was an isomorphism, it follows that β represents the non-trivial element of $\pi_1(\mathbb{P}(\mathbb{R}^7), z_1)$.

On the other hand the path $\tilde{\beta} : I \rightarrow \mathbb{R}^7 \setminus \{0\}$ is a lifting of the loop β to $\mathbb{R}^7 \setminus \{0\}$ which starts at the point $\tilde{z}_1 \in \mathbb{R}^7 \setminus \{0\}$ and ends at the point $\tilde{z}_2 = (-1)^d \tilde{z}_1$. Since β represents the non-trivial element of $\pi_1(\mathbb{P}(\mathbb{R}^7), z_1)$, it follows from Lemma 2.1 that $(-1)^d < 0$ and, thus, the degree $d = \deg(\Phi)$ is odd.

That completes the proof of Proposition 2.1. \square

Remark 2.3. The space $\mathbb{R}^7 \setminus \{0\}$ is homotopy equivalent to the 6-dimensional sphere S^6 . Its homology group $H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z})$ with coefficients in \mathbb{Z} in dimension 6 is isomorphic to the group of integers, $H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z}) \cong \mathbb{Z}$. The continuous mapping $\Phi : \mathbb{R}^7 \setminus \{0\} \rightarrow \mathbb{R}^7 \setminus \{0\}$ induces a group homomorphism $\Phi_* : H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z}) \rightarrow H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z})$ which is given by multiplication by an integer usually called the (topological) degree of the map Φ . Since the map $\eta_{\mathbb{P}} : \mathbb{P}(\mathbb{R}^7) \rightarrow \mathbb{P}(\mathbb{R}^7)$ is a homeomorphism and the algebraic degree $d = \deg(\Phi)$ in the sense of this paper is odd, it is rather easy to see that also the mapping $\Phi : \mathbb{R}^7 \setminus \{0\} \rightarrow \mathbb{R}^7 \setminus \{0\}$ is a homeomorphism. It follows that $\Phi_* : H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z}) \rightarrow H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z})$ is an isomorphism and, hence, its topological degree can only be equal to ± 1 . Thus the topological degree of the map Φ and its algebraic degree $d = \deg(\Phi)$ in the sense of [7] and of the present paper are different notions, at least when the algebraic degree $d = \deg(\Phi)$ is equal to 3 or 5.

Now let A be an 8-dimensional real quadratic division algebra. Since A is a quadratic algebra, Frobenius's lemma [10, 13] applies. It asserts that the set $V = \{v \in A \setminus \mathbb{R}1 \mid v^2 \in \mathbb{R}1\} \cup \{0\}$ of all purely imaginary elements in A is a hyperplane in A , such that $A = \mathbb{R}1 \oplus V$. This Frobenius

decomposition of A gives rise to the \mathbb{R} -linear maps $\varrho : A \rightarrow \mathbb{R}$ and $\iota : A \rightarrow V$ such that $a = \varrho(a)1 + \iota(a)$ for all $a \in A$. The induced algebra structure on V , i.e. the bilinear map $\eta : V \times V \rightarrow V$, $\eta(v, w) = \iota(vw)$, is anticommutative. Therefore it may be identified with the linear map $\eta : V \wedge V \rightarrow V$, $\eta(v \wedge w) = \iota(vw)$. Since A is a division algebra, this linear map η is dissident [14]. Any choice of a basis in V identifies η with a dissident map η on \mathbb{R}^7 , and the degree of η does not depend on the chosen basis. It is therefore justified to define the *degree* of an 8-dimensional real quadratic division algebra A by $\deg(A) := \deg(\eta)$.

Corollary 2.4. *The degree of any 8-dimensional real quadratic division algebra is 1, 3, or 5.*

Proof. Let $A \in \mathcal{D}_8^q$. Then $\deg(A) = \deg(\eta) = d$, where $1 \leq d \leq 5$ by [7, Theorem 2.4], and $\deg(\eta)$ is odd by Proposition 2.1. \square

Corollary 2.5. *The category \mathcal{D}_8^q decomposes into its non-empty full subcategories \mathcal{D}_8^{q1} , \mathcal{D}_8^{q3} , and \mathcal{D}_8^{q5} .*

Proof. Corollary 2.4 states that the object class of \mathcal{D}_8^q is the disjoint union of the object classes of \mathcal{D}_8^{q1} , \mathcal{D}_8^{q3} , and \mathcal{D}_8^{q5} .

Let $f : A \rightarrow A'$ be a morphism in \mathcal{D}_8^q . The algebra structures on A and A' induce dissident maps η and η' on the purely imaginary hyperplanes V and V' of A and A' , respectively. Now f is injective because A is a division algebra, and furthermore even bijective because $\dim(A) = \dim(A')$ is finite. So f is an isomorphism of algebras. It induces an isomorphism of dissident maps $\sigma : \eta \rightarrow \eta'$, i.e. a linear bijection $\sigma : V \rightarrow V'$ satisfying $\sigma\eta(v \wedge w) = \eta'(\sigma(v) \wedge \sigma(w))$ for all $v, w \in V$. We conclude with [7, Proposition 3.1] that $\deg(\eta) = \deg(\eta')$, hence $\deg(A) = \deg(A')$. Thus f is a morphism in \mathcal{D}_8^{qd} for some $d \in \{1, 3, 5\}$.

Altogether this proves the decomposition $\mathcal{D}_8^q = \mathcal{D}_8^{q1} \sqcup \mathcal{D}_8^{q3} \sqcup \mathcal{D}_8^{q5}$ of the category \mathcal{D}_8^q . Non-emptiness of its blocks \mathcal{D}_8^{q1} , \mathcal{D}_8^{q3} , and \mathcal{D}_8^{q5} follows from [7, Section 6], where objects are constructed for each of them. \square

3. On the structure of \mathcal{D}_8^{q1} , \mathcal{D}_8^{q3} , and \mathcal{D}_8^{q5}

Corollary 2.5 reduces the problem of understanding the structure of \mathcal{D}_8^q to the problem of understanding the structures of \mathcal{D}_8^{q1} , \mathcal{D}_8^{q3} , and \mathcal{D}_8^{q5} . We proceed to summarize the present state of knowledge regarding the latter problem.

A *dissident triple* (V, ξ, η) consists of a (finite-dimensional) Euclidean space $V = (V, \langle \cdot, \cdot \rangle)$, a linear form $\xi : V \wedge V \rightarrow \mathbb{R}$, and a dissident map $\eta : V \wedge V \rightarrow V$. The class of all dissident triples forms a category \mathcal{V} whose morphisms $\varphi : (V, \xi, \eta) \rightarrow (V', \xi', \eta')$ are orthogonal linear maps $\varphi : V \rightarrow V'$ satisfying $\xi = \xi'(\varphi \wedge \varphi)$ and $\varphi\eta = \eta'(\varphi \wedge \varphi)$. If (V, ξ, η) is a dissident triple, then the vector space $\mathcal{F}(V, \xi, \eta) = \mathbb{R} \times V$, equipped with the multiplication

$$(\alpha, v)(\beta, w) = (\alpha\beta - \langle v, w \rangle + \xi(v \wedge w), \alpha w + \beta v + \eta(v \wedge w)),$$

is a real quadratic division algebra. If $\varphi : (V, \xi, \eta) \rightarrow (V', \xi', \eta')$ is a morphism of dissident triples, then the linear map

$$\mathcal{F}(\varphi) : \mathcal{F}(V, \xi, \eta) \rightarrow \mathcal{F}(V', \xi', \eta'), \quad \mathcal{F}(\varphi)(\alpha, v) = (\alpha, \varphi(v))$$

is a morphism of real quadratic division algebras. It is well known [5,9] that Osborn's theorem [14] can be rephrased in the language of categories and functors as follows.

Theorem 3.1. *The functor $\mathcal{F} : \mathcal{V} \rightarrow \mathcal{D}^q$ is an equivalence of categories.*

For each $d \in \{1, 3, 5\}$ we denote by \mathcal{V}_7^d the full subcategory of \mathcal{V} formed by all dissident triples (V, ξ, η) satisfying $\dim(V) = 7$ and $\deg(\eta) = d$. Then the equivalence of categories $\mathcal{F} : \mathcal{V} \rightarrow \mathcal{D}^q$ induces equivalences of categories $\mathcal{F}_7^d : \mathcal{V}_7^d \rightarrow \mathcal{D}_8^{qd}$ for all $d \in \{1, 3, 5\}$. The category \mathcal{V}_7^1 admits an equivalent description entirely in terms of matrices, which we proceed to recall.

The octonion algebra \mathbb{O} is well known to be quadratic. Hence it has Frobenius decomposition $\mathbb{O} = \mathbb{R}1 \oplus V$, and thereby it determines \mathbb{R} -linear maps $\varrho : \mathbb{O} \rightarrow \mathbb{R}$ and $\iota : \mathbb{O} \rightarrow V$ such that $a = \varrho(a)1 + \iota(a)$ for all $a \in \mathbb{O}$. The symmetric \mathbb{R} -bilinear form

$$\mathbb{O} \times \mathbb{O} \rightarrow \mathbb{R}, \quad \langle x, y \rangle = 2\varrho(x)\varrho(y) - \frac{1}{2}\varrho(xy + yx)$$

is well known to be positive definite, thus equipping \mathbb{O} with the structure of a Euclidean space. Every algebra automorphism $\alpha \in \text{Aut}(\mathbb{O})$ fixes the unity 1 of \mathbb{O} and is orthogonal. Since $1^\perp = V$, it induces an orthogonal linear endomorphism $\alpha_V \in \text{O}(V)$. The map $\nu : \text{Aut}(\mathbb{O}) \rightarrow \text{O}(V)$, $\nu(\alpha) = \alpha_V$ is an injective group homomorphism. Choosing an orthonormal basis in V , the subgroup $\nu(\text{Aut}(\mathbb{O})) < \text{O}(V)$ is identified with a subgroup of $\text{O}(7)$, which classically is denoted by \mathbb{G}_2 . Simultaneously, the dissident map

$$\eta : V \wedge V \rightarrow V, \quad \eta(v \wedge w) = \iota(vw)$$

is identified with a *vector product map* $\mathbb{R}^7 \wedge \mathbb{R}^7 \rightarrow \mathbb{R}^7$, $v \wedge w \mapsto v \times w$. Now denote by $\mathbb{R}^{7 \times 7}$ the set of all real 7×7 -matrices, and by $\mathbb{R}_{\text{ant}}^{7 \times 7}$, $\mathbb{R}_{\text{pds}}^{7 \times 7}$, $\mathbb{R}_{\text{spds}}^{7 \times 7}$ the subsets of $\mathbb{R}^{7 \times 7}$ consisting of all matrices which are antisymmetric, positive definite symmetric, and positive definite symmetric of determinant 1, respectively. We view the matrix quadruple set

$$\mathcal{Q} = \mathbb{R}_{\text{ant}}^{7 \times 7} \times \mathbb{R}_{\text{ant}}^{7 \times 7} \times \mathbb{R}_{\text{pds}}^{7 \times 7} \times \mathbb{R}_{\text{spds}}^{7 \times 7}$$

as the object set of a groupoid \mathcal{Q} whose morphisms

$$S : (A, B, C, D) \rightarrow (A', B', C', D')$$

are the orthogonal matrices $S \in \mathbb{G}_2$ which satisfy

$$(SAS^t, SBS^t, SC S^t, SDS^t) = (A', B', C', D').$$

Then the groupoid \mathcal{Q} and the category \mathcal{V}_7^1 are related as follows. (For proofs see [7, Section 5].)

Theorem 3.2.

- (i) If $(A, B, C, D) \in \mathcal{Q}$ then $\mathcal{G}(A, B, C, D) = (\mathbb{R}^7, \xi, \eta)$, with $\xi(v \wedge w) = v^t A w$ and $\eta(v \wedge w) = (B + C)D(Dv \times Dw)$, is in \mathcal{V}_7^1 .
- (ii) If $S : (A, B, C, D) \rightarrow (A', B', C', D')$ is a morphism in \mathcal{Q} then $\mathcal{G}(S) : \mathcal{G}(A, B, C, D) \rightarrow \mathcal{G}(A', B', C', D')$, given by $\mathcal{G}(S)(v) = Sv$, is a morphism in \mathcal{V}_7^1 .
- (iii) The functor $\mathcal{G} : \mathcal{Q} \rightarrow \mathcal{V}_7^1$ is an equivalence of categories.

Composing the functors \mathcal{G} and \mathcal{F}_7^1 to $\mathcal{H} = \mathcal{F}_7^1 \mathcal{G}$, we arrive at the following explicit description of the category \mathcal{D}_8^{q1} entirely in terms of matrices.

Corollary 3.3. *The functor $\mathcal{H} : \mathcal{D} \rightarrow \mathcal{D}_8^{q1}$ is an equivalence of categories. It is given on objects by $\mathcal{H}(A, B, C, D) = \mathbb{R} \times \mathbb{R}^7$, with multiplication $(\alpha, v)(\beta, w) = (\alpha\beta - v^t w + v^t A w, \alpha w + \beta v + (B + C)D(Dv \times Dw))$, and on morphisms by $\mathcal{H}(S)(\alpha, v) = (\alpha, Sv)$.*

On the other hand, we do not know any description of the categories \mathcal{V}_7^3 or \mathcal{V}_7^5 entirely in terms of matrices. Indeed, the 7-dimensional *dissident algebras* (V, η) of degree 3 or 5 which are inherent in the objects (V, ξ, η) of \mathcal{V}_7^3 or \mathcal{V}_7^5 respectively seem hardly to be understood at present.

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